Graphs With Same Adjacency & Incidence Matrix.

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Abstract

[1] The content is a solution of the problem appeared in the American Mathematical Monthly[Problem No. 10967, October 2002, Page 759].
The question is:
Let A be the adjacency matrix of a simple graph G.
(a) For which G is A the incidence matrix of a simple graph?
(b) For which G is the A the incidence matrix of a graph isomorphic to G?

1 Preliminaries

Let A be the adjacency matrix of a simple labeled graph G. A is of order $n \times n$ where n is the number of vertices of graph G. The $(i, j)^{th}$ entry of A is 1 if and only if v_i and v_j are adjacent vertices. Otherwise the entry is zero. Clearly, A is symmetric with all diagonal entries zero.

An incidence matrix of a simple labeled graph is of order $n \times m$ where m is the number of edges. Here $(i, j)^{th}$ entry is 1 if and only if the j^{th} edge is

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[1]The solution had already been acknowledged in The American Mathematical Monthly, Vol. 111, No. 5 (May, 2004), p. 443. The article is the revised version of the solution solution submitted. In particular, the statement of the theorem 2.1 is corrected as per remarks 3.3(4).

incident on the vertex v_i. Hence in each column of an incidence matrix there are exactly two unit entries. The number of $1'$s in the i^{th} row is the degree of the vertex v_i.

2 Answer of the first question

First, suppose that G is a simple labeled connected graph.

Theorem 2.1. *Let G be a simple connected graph and let A be its adjacency matrix. If A is an incidence matrix of some simple graph G' then G is regular of degree 2(i.e., G is a cycle). The converse is true if $n \neq 4$.*

Proof: The adjacency matrix A to be the incidence matrix of some simple graph, it is essential that $n = m$ and each column has exactly two unit entries. Since the number of 1's in the i^{th} row of an incidence matrix is the degree of the vertex v_i, G must be regular of degree 2.
Claim:G is a cycle of length n.
We have the result:[[1], Exercise 4.4, page 42.]
The following four statements are equivalent for a graph G with p vertices and q edges.
(1) G is unicyclic.
(2) G is connected and $p = q$.
(3) For some line x of G, the graph $G - \{x\}$ is a tree.
(4) G is connected and the set of lines of G which are not bridges form a cycle.

Therefore, G is unicyclic. As, G is regular connected simple graph of degree 2, it is easy to see that G is a cycle. Hence the claim.

As in [2], for a graph G with 4 vertices G' will not be a simple graph. In fact in this case G' is not connected, It consists of two loops. Otherwise the converse is true and is clear from the definition of an incidence matrix.

3 Answer of the second question

Denote, the simple graph for which A is the incidence matrix, by G'. Let G be the cycle $v_1 v_2 v_3 v_4 \cdots v_{n-1} v_n v_1$ where v_i are vertices. Hence $A = (a_{ij})$ is such that

$$\begin{aligned} a_{i,i+1} &= a_{i+1,i} \\ &= 1 \quad i = 1, 2, \ldots, n-1 \\ a_{1,n} &= a_{n,1} \\ &= 1 \\ a_{ij} &= 0 \quad \text{otherwise} \end{aligned}$$

Observations 3.1. 1. v_i is adjacent to v_s and v_t in G(i, s, t are all distinct) if and only if v_s and v_t are adjacent in G'.
2. Each row of A(as it is symmetric as the adjacency matrix of G) has exactly two unit entries, therefore, G' is also regular of degree 2.

Theorem 3.2. *Let G be a cycle. Then, $G \cong G'$ if and only if n is odd.*

Proof: If n is odd , it is easy to show that, by the observation 3.1(1), G' is the cycle:

$$v_1 v_3 v_5 v_7 \cdots v_n v_2 v_4 v_6 v_8 \cdots v_{n-1} v_1$$

Hence $G \cong G'$.
Now, suppose n is even.
In this case $n-1$ is odd. Therefore

$$v_1 v_3 v_5 v_7 \cdots v_{n-1} v_1$$

$$v_n v_2 v_4 v_6 v_8 \cdots v_{n-2} v_n$$

are two cycles in G' if $n > 4$. Thus $G \ncong G'$.

Remarks 3.3. 1. One can easily check that $f : G \to G'$ such that

$$\begin{aligned} f(v_i) &= v_{2i-1} & i &= 1, 2, \ldots, \tfrac{n+1}{2} \\ f(v_j) &= v_{2j-(n+1)} & j &= \tfrac{n+1}{2} + 1, \tfrac{n+1}{2} + 2, \ldots, n. \end{aligned}$$

is a graph isomorphism, where n is odd and $n \geq 3$.
2. By observations 3.1(1), if v_s and v_t are adjacent in G then v_i, v_s, v_t each has degree 2 and we have the case $n = 3$. But, in general, v_s and v_t are not adjacent in G when $n > 3$.
3. Consider the graph for $n = 3$ labeled as below:

$$e_1 \equiv \{v_2, v_3\}, e_2 \equiv \{v_1, v_3\}, e_1 \equiv \{v_1, v_2\}$$

In this case, we have same adjacency matrix and incidence matrix i.e., G and G' are identical.
4. The construction of G' also shows that for $n = 4$ G' is not simple.
5. The construction of G' in the theorem 3.2 and the observations clearly show that for $n > 3$, G and G' are isomorphic if n is odd but not identical.

Remarks 3.4. 1. If G is not connected in the theorem 2.1, then each component is a cycle.
2. If G is a simple regular graph of degree 2 with k components, it is evident that the theorem 3.2 must be true for each component.

References:

[1] Haray F., Graph Theory, Narosa Publishing House.

[2] The American Mathematical Monthly, Vol. 111(5), May 2004, p. 443.